Intro video: Section 2.6 part 2
Tricky limits at infinity

Math F251X: Calculus I

Example: $\lim\limits_{x\to\infty} \dfrac{2x-5}{x^{3/2}-8}$

Not a polynomial down here

$$= \lim_{x\to\infty} \left(\frac{2x-5}{x^{3/2}-8}\right)\frac{1/x^{3/2}}{1/x^{3/2}} = \lim_{x\to\infty}\frac{2x^{2/2}\cdot x^{-3/2} - 5/x^{3/2}}{1 - 8/x^{3/2}}$$

$$= \lim_{x\to\infty}\frac{2x^{-1/2} - 5/x^{3/2}}{1 - 8/x^{3/2}} = \lim_{x\to\infty}\frac{2/\sqrt{x} - 5/x\sqrt{x}}{1 - 8/x\sqrt{x}}$$

As $x \to \infty$, $\sqrt{x} \to \infty$

$$= \frac{\lim\limits_{x\to\infty} 2/\sqrt{x} - \lim\limits_{x\to\infty} 5/x^{3/2}}{1 - \lim\limits_{x\to\infty} 8/x^{3/2}}$$

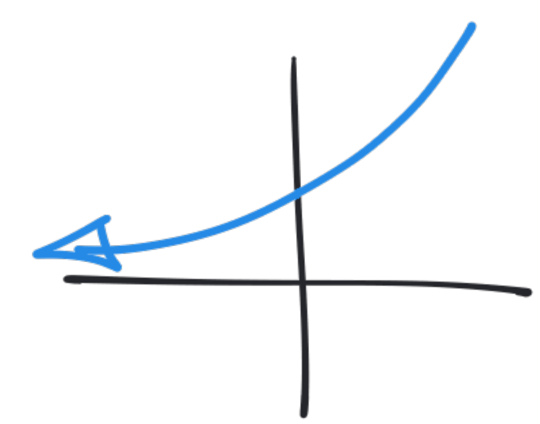
$$= \frac{0-0}{1-0} = 0.$$

Example ① $\lim\limits_{x\to\infty}\left(\dfrac{2e^x}{8-(\sqrt{5})e^x}\right)\dfrac{1/e^x}{1/e^x} = \lim\limits_{x\to\infty}\dfrac{2}{8/e^x-\sqrt{5}}$

If we "sub in" ∞ (ugh, ∞ is not a #) we get type $\dfrac{\infty}{\infty}$

$$= \frac{2}{\left(\lim\limits_{x\to\infty} 8/e^x\right)-\sqrt{5}} = \frac{2}{0-\sqrt{5}} = \frac{-2}{\sqrt{5}}$$

② $\lim\limits_{x\to-\infty}\dfrac{2e^x}{8-(\sqrt{5})e^x} = \dfrac{2\lim\limits_{x\to-\infty}e^x}{8-\sqrt{5}\lim\limits_{x\to-\infty}e^x}$



$$= \frac{0}{8} = 0$$

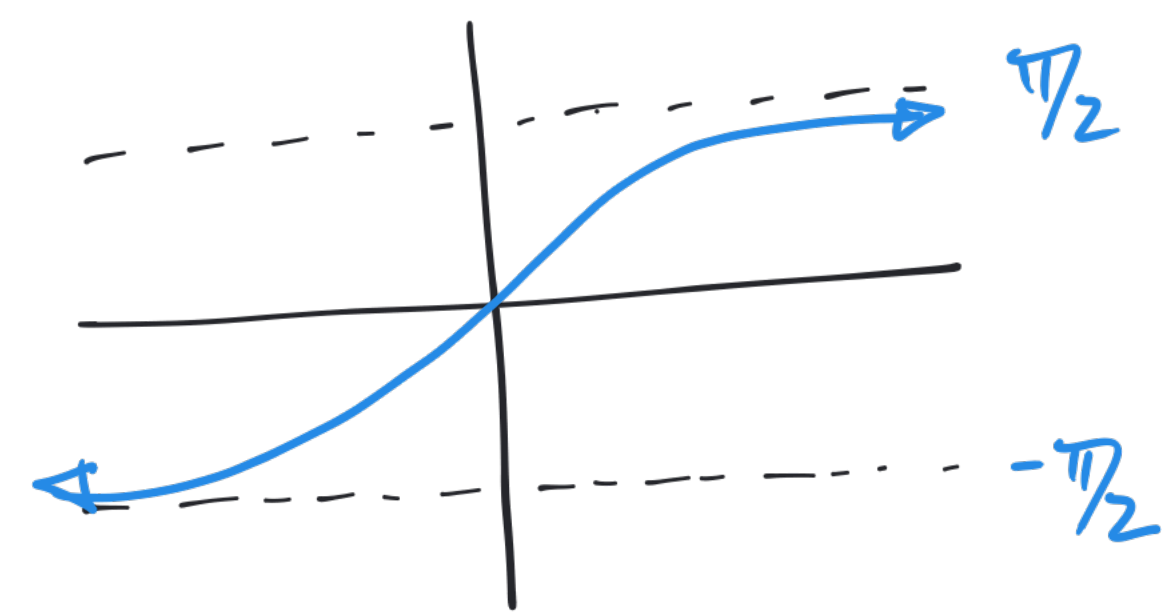
Example $\lim_{x \to -\infty} e^{\arctan(x)}$

$e^{x} = \exp(x)$

$= \lim_{x \to -\infty} \exp(\arctan(x))$

$\exp(x)$ is a continuous function!

$= \exp\left(\lim_{x \to -\infty} \arctan(x) \right)$



$= \exp(-\pi/2)$

$= e^{-\pi/2} = 1/e^{\pi/2}$

Example: $\displaystyle\lim_{x\to-\infty} \sqrt[3]{x} - x^3$

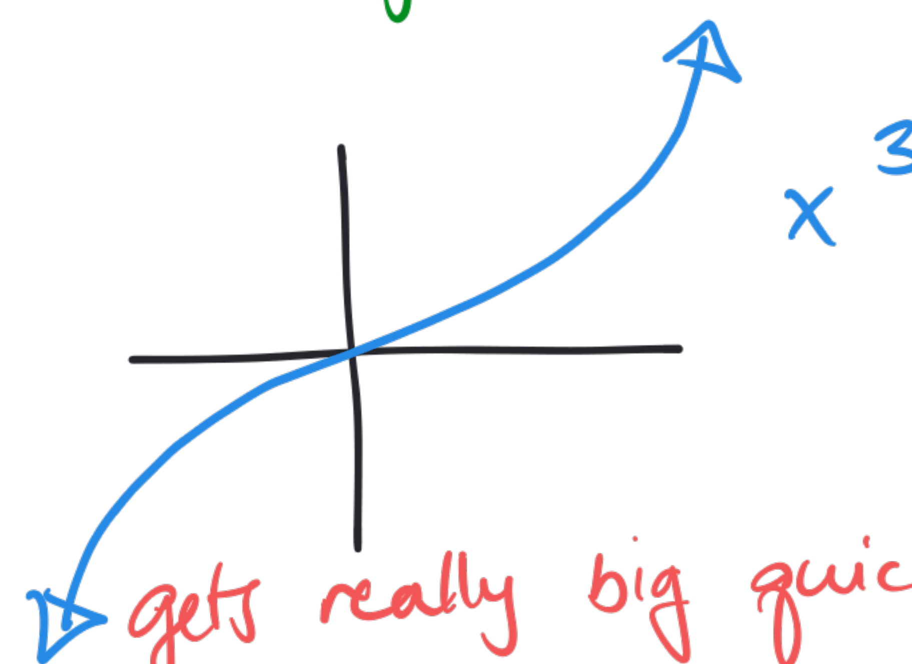
type "$\infty - \infty$"

$$= \lim_{x\to-\infty} \left(\sqrt[3]{x} - x^3\right)\left(\frac{\sqrt[3]{x} + x^3}{\sqrt[3]{x} + x^3}\right)$$

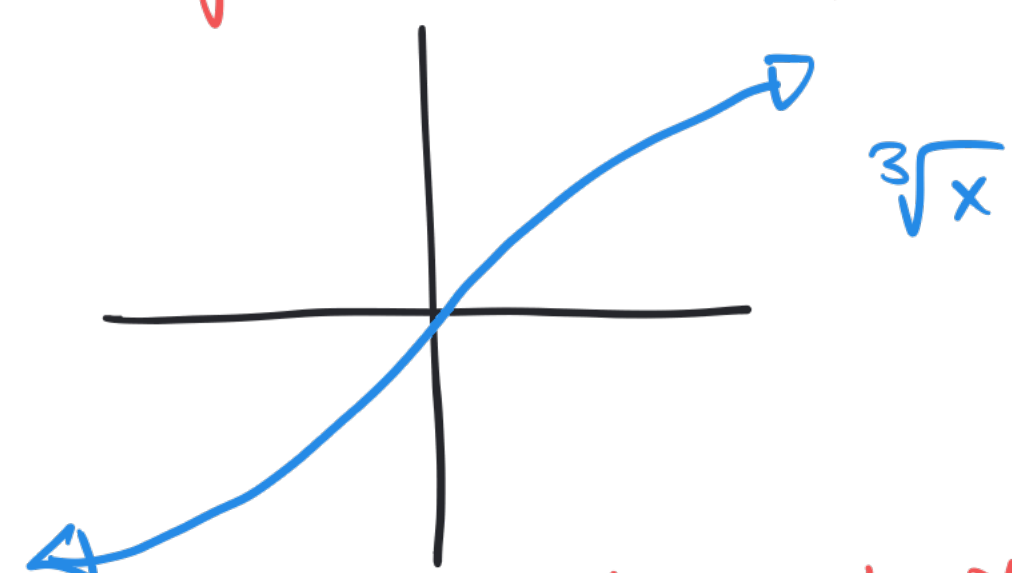
$$= \lim_{x\to-\infty} \left(\frac{x - x^6}{\sqrt[3]{x} + x^3}\right)\left(\frac{1/x^3}{1/x^3}\right)$$

$$= \lim_{x\to-\infty} \frac{1/x^2 - x^3}{\frac{x^{1/3}}{x^{27/3}} + 1}$$

$$= \lim_{x\to\infty} \frac{1/(-x)^2 - (-x)^3}{1/(-x)^{26/3} + 1} = \lim_{x\to\infty} \frac{\overbrace{1/x^2}^{\to 0} + \overbrace{x^3}^{\to\infty}}{\underbrace{1/x^{26/3}}_{\to 0} + 1} = \infty$$



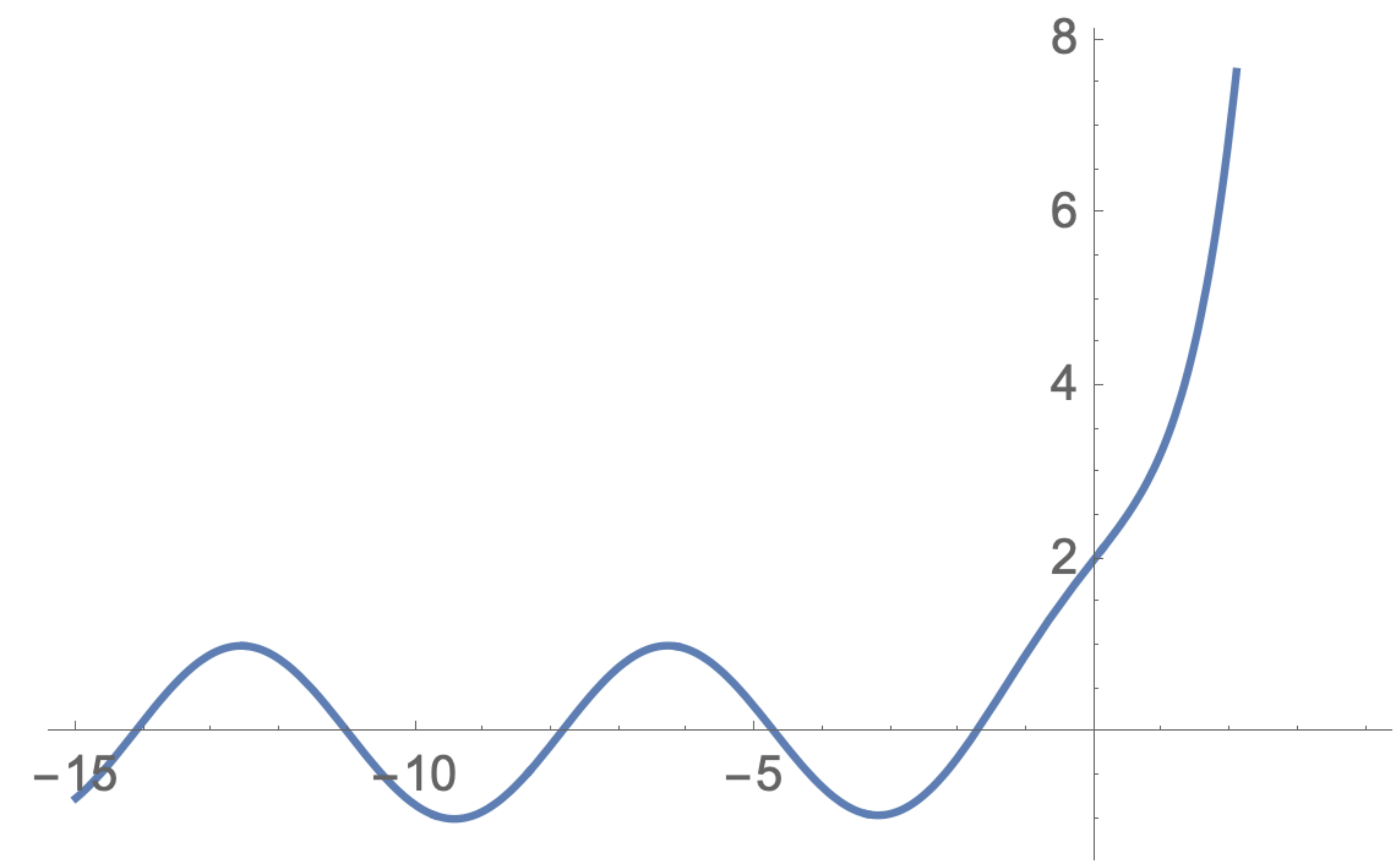
gets really big quickly



gets really big slowly

Example: $f(x) = e^x + \cos(x)$.

```
Plot[Exp[x] + Cos[x], {x, -15, 4}]
```



$$\lim_{x \to \infty} e^x + \cos(x) = ?$$

Observe $-1 \leq \cos(x) \leq 1$.

$$e^x - 1 \leq e^x + \cos(x) \leq e^x$$

So $$\lim_{x \to \infty} e^x - 1 \leq \underbrace{\lim_{x \to \infty} e^x + \cos(x)}_{\to \infty \text{ as } x \to \infty}$$

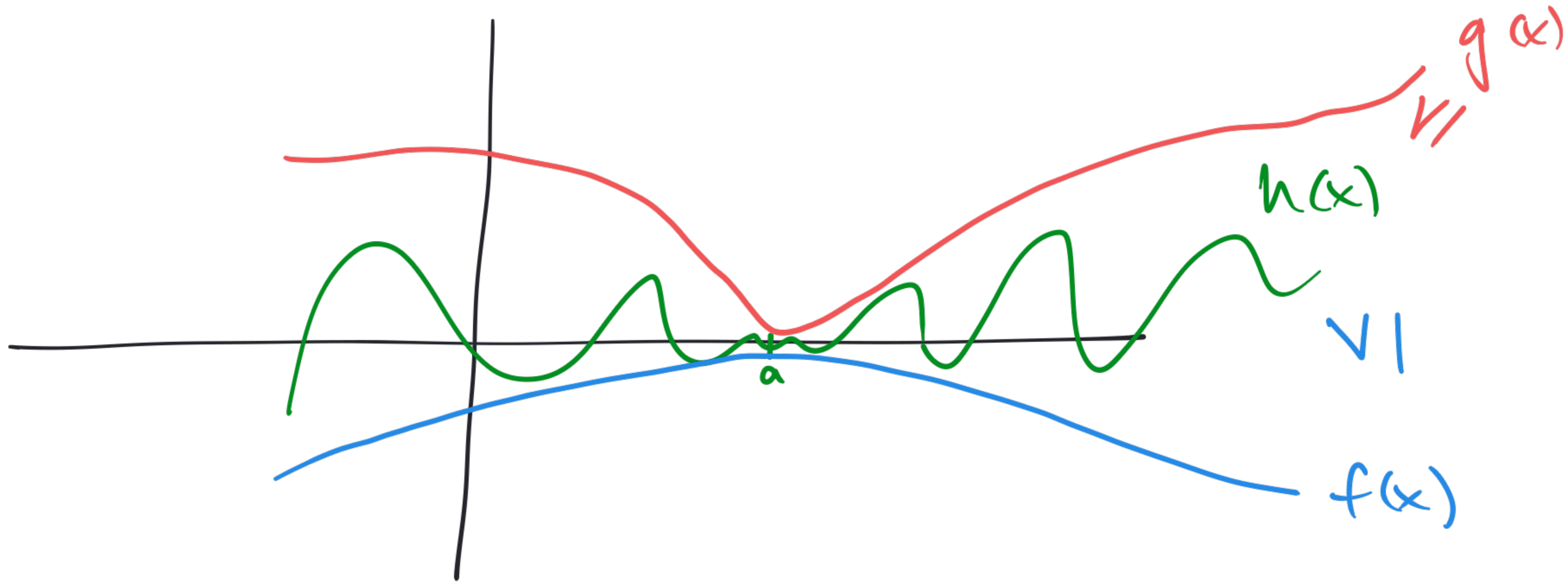
$\hookrightarrow$ goes to ∞ as $x \to \infty$

So $\lim_{x \to \infty} e^x + \cos(x) = \infty$.

However, $$\lim_{x \to -\infty} e^x + \cos(x) = \lim_{x \to -\infty} e^x + \lim_{x \to -\infty} \cos(x)$$

$$= 0 + \lim_{x \to -\infty} \cos(x) = \text{DNE}$$

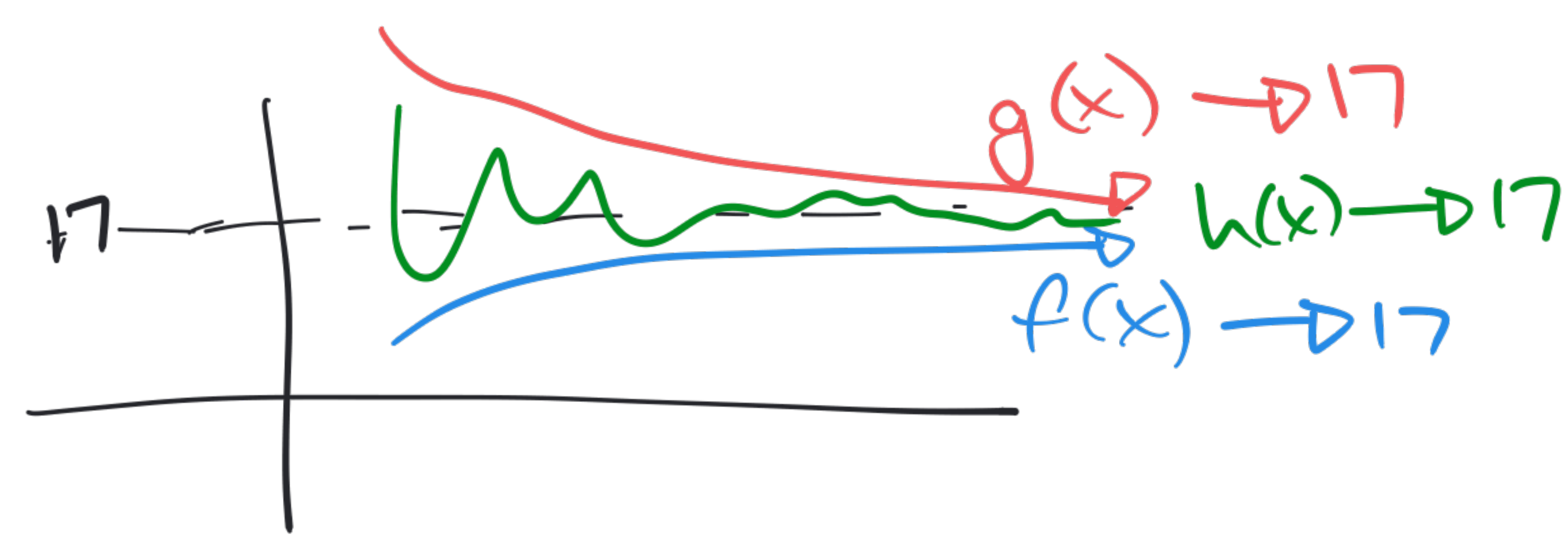
The Squeeze Theorem



Suppose near a, $f(x) \le h(x) \le g(x)$. Then if

$\lim_{x \to a} f(x) = L = \lim_{x \to a} g(x)$, then $\lim_{x \to a} h(x) = L$.

Also works as $x \to \infty$



Squeeze example

$$h(x) = \frac{\sin(x)}{e^x} + 17$$

$$-1 \leq \sin(x) \leq 1 \qquad \text{As } x \to \infty, \ \frac{1}{e^x} \to 0$$

$$-\frac{1}{e^x} \leq \frac{\sin(x)}{e^x} \leq \frac{1}{e^x} \implies$$

$$-\frac{1}{e^x} + 17 \leq \frac{\sin(x)}{e^x} + 17 \leq \frac{1}{e^x} + 17$$

So $\lim_{x \to \infty} 17 - \frac{1}{e^x} \leq \lim_{x \to \infty} \frac{\sin(x)}{e^x} + 17 \leq \lim_{x \to \infty} \frac{1}{e^x} + 17$

$$17 \leq \lim_{x \to \infty} \frac{\sin(x)}{e^x} + 17 \leq 17$$

By the squeeze theorem, $\lim_{x \to \infty} \frac{\sin(x)}{e^x} + 17 = 17.$